Loop Filter Design for Phase-Locked Loops with Delay: A Multi-objective Control Approach

Y.S. Chou, Y.C. Chen, W.L. Mao and F.R. Chang

Abstract—Two loop filter design methods for phase-locked loops (PLLs) with loop delay are presented, which employs multi-objective control technique to deal with the various design objectives: small noise bandwidth and large gain and phase margins, etc. Trade-off among the conflicting objectives is made via recently developed convex optimization skill in conjunction with appropriate adjustment of certain design parameters. By applying first-order Padé approximation for the delay term, one of the design methods has the advantage that almost all except one of the filter poles can be assigned a priori. On the other hand, through applying a special coordinate transformation technique, a new PI form loop filter design is also presented. It is noted that both of the proposed methods are applicable to PLL of any order. Numerical simulation on nonlinear PLL model is performed which demonstrates the effectiveness of the proposed methods.

Index Terms—PLL, time delay, PI form filter, multi-objective control, linear matrix inequality (LMI).

I. INTRODUCTION

SINCE its invention, the PLL principle [1], [2], [3], [4] has been used in a wide spread of applications, such as carrier tracking [5], timing recovery [6], and servo control [7], [8], etc [9]. From the system's point of view, PLL is essentially a nonlinear system. The design of PLL with a sinusoidal-characteristic phase detector using Lyapunov redesign technique can be found in [10], [11]. However, there is a difficulty in applying the proposed method to high-order loops. Besides, large decision delays frequently exist in modern receivers, which might degrade the performance of the systems, leading to smaller damping, larger settling time, reduction of the phase margin and even more causing to instability. Recently, a method based on linear model approximation was proposed which designs a loop filter that minimizes the phase error variance with guaranteed gain margin and phase margin in the presence of phase detector gain uncertainty and constant loop delay [12]. A complicated design procedure was provided via applying coprime

factorization control theory [13] and quantitative feedback theory (QFT) [14]. In many cases, PI form filters which have all of the poles at the origin are favorable for some advantages [3] they brought in: infinite hold range and pull-in range (theoretically), and good phase tracking capability. However, the filters obtained by the approach just mentioned usually have poles not at the origin. Having noticed this point, an independent design procedure of PI form filter has also been addressed in [12] for second-order loop. Nevertheless, it can be checked that it is hard to extend the method therein to higher order loops. In [15] the same problem for PLLs with polytopic uncertainties is studied using another approach. Similarly, the filters obtained are generally not of PI form.

In this paper, two types of loop filters are considered: the general form filters (in which not all of the poles are locating at the origin) and the PI form filters. The design of the class of general form filters is transformed into multi-objective static state feedback synthesis problem, where the various design objectives: small noise bandwidth, good transient response [16], and large gain and phase margins [16] are all taken into consideration. Trade-off among them is made via linear matrix inequality (LMI) optimization [17], [18], [19], [20] in conjunction with appropriate adjustment of certain design parameters. Similarly, the synthesis of the class of PI form filters is formulated as a mixed H_2/H_∞ control problem. A special coordinate transformation technique [21] is employed to deal with the problem. The paper is organized as follows. In Section II, the preliminaries and problem statement are given. In Section III, the design of the general form loop filters is presented. Section IV provides the PI form filter design. Section V shows the simulation results. Finally, Section VI gives the conclusion. The detailed definitions of the H_2 and H_∞ norm of stable transfer functions can be found in [22]. Throughout this paper, a signal in time domain and frequency domain are denoted by lower case and upper case, respectively.

II. PRELIMINARIES AND PROBLEM STATEMENT

A. Basic model of PLL

Fig.1 PLL schematic model

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The PLL model used here is depicted in Fig. 1, which consists of a phase detector, low-pass loop filter $F(s)$, and voltage controlled oscillator (VCO). The inputs to the phase detector are the two signals: the sum of the carrier and noise $n(t)$ that is stationary, Gaussian, bandpass and zero mean [4], i.e.,

$$u(t) = \sqrt{2}A_d \sin(\omega_0 t + w_\theta(t)) + n(t),$$

and the VCO output

$$\hat{u}(t) = \sqrt{2} \cos(\omega_0 t + \hat{w}_\theta(t)).$$

The phase detector produces, assuming the high frequency term is eliminated by the low-pass filter, the output signal

$$u_d(t) = A_d [\sin(w_\theta - \hat{w}_\theta) + n'(t)]$$

where n' represents the net effect caused by the noise n . For small phase errors and small input phase disturbance $w_n(t)$, the PLL can be further approximated by the linear model as depicted in Fig. 2.

Fig. 2 PLL linear model

B. The goal

In view of Fig. 2, the goal of this paper is to design a filter $F(s)$ to achieve the following objectives:

- (i) Closed-loop stability,
- (ii) Perfect asymptotical tracking (i.e. $e(\infty) = 0$) subject to the deterministic test signals $w_\theta(t) = t^k$, $k = 0, 1, 2, \dots, m$ when $w_n = 0$,
- (iii) Good transient response (i.e., small settling time, small overshoot, etc),
- (iv) Noise attenuation (assuming w_n to be white noise),
- (v) Large stability margin (i.e. gain margin (GM), phase margin (PM)).

Note that the variance of VCO output phase is given by

$$\sigma_{\hat{w}_\theta}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |T_{\hat{w}_\theta w_n}(j\omega)|^2 \Phi_{w_n}(\omega) d\omega$$

where $T_{\hat{w}_\theta w_n}$ represents the transfer function from w_n to $\hat{w}_\theta$, and $\Phi_{w_n}(\omega)$ is the power spectral density (PSD) function of the noise w_n . If w_n is white Gaussian, i.e., $\Phi_{w_n}(\omega) = N_0$, then

$$\sigma_{\hat{w}_\theta}^2 = N_0 \|T_{\hat{w}_\theta w_n}\|_2^2 = 2N_0 B_n$$

where B_n (Hz) denotes the noise bandwidth of $T_{\hat{w}_\theta w_n}$. It is clear that small noise bandwidth B_n leads to small variance of the VCO output phase. In terms of the loop filter design, this

can be achieved by minimizing the H_2 norm of the closed-loop system $T_{\hat{w}_\theta w_n}$ over all the stabilizing filters.

On the other hand, the open-loop response should have enough gain and phase margins in order to guarantee good relative stability and well-behaved closed-loop response. It is mentioned in [12], [14] that these margins can be determined by the H_∞ norm upper bounds of certain closed-loop system. Specifically, let $L(s)$ denote the open loop transfer function of Fig. 2. Assume a constant γ or alternatively a constant δ satisfies the following H_∞ norm condition:

$$\left| \frac{L(j\omega)}{1+L(j\omega)} \right| < \gamma \quad \left(\text{resp.} \quad \left| \frac{1}{1+L(j\omega)} \right| < \delta \right)$$

(1)

A pair of lower bounds of the gain and phase margins is then determined by the formulas:

$$20 \log \frac{\gamma+1}{\gamma} \text{ dB}, \quad 2 \sin^{-1} \frac{1}{2\gamma} \text{ deg} \quad \left(\text{resp.} \quad 20 \log \frac{\delta}{\delta-1} \text{ dB}, \quad 2 \sin^{-1} \frac{1}{2\delta} \text{ deg} \right) \quad (2)$$

Note that condition (1) is equivalent to a H_∞ norm bound constraint of the closed-loop system $T_{\hat{w}_\theta w_n}$. Thus minimizing the H_∞ norm of the closed-loop system $T_{\hat{w}_\theta w_n}$ over all the stabilizing filters is expected to increase the gain and phase margins.

C. Useful Analysis LMIs

In the following, Lemma 1 [18] and Lemma 2 [18] present the H_2 and H_∞ norm constraints of a LTI system in terms of LMIs, respectively. Lemma 3 [19] and lemma 4 [19] state that all the eigenvalues of a square matrix lie in a prescribed region if and only if certain LMIs are feasible.

Lemma 1 [18]: Given a LTI system $H(s) = C(sI - A)^{-1}B$, and a positive value ν . The following statements are equivalent.

- (i) the LTI system $H(s)$ is stable and $\|H\|_2 < \nu$.
- (ii) there exist matrices $P = P^T$ and $Q = Q^T$ such that the following LMIs hold,

$$\begin{pmatrix} AP + PA^T & B \\ B^T & -I \end{pmatrix} < 0, \quad \begin{pmatrix} P & (CP)^T \\ CP & Q \end{pmatrix} > 0, \\ \text{Tr}(Q) < \nu^2.$$

Lemma 2 [18]: Given a LTI system $H(s) = C(sI - A)^{-1}B + D$, and a positive value γ . The following statements are equivalent.

- (i) The LTI system $H(s)$ is stable and $\|H\|_\infty < \gamma$.
- (ii) There exists a matrix $P = P^T$ such that the following LMIs hold,

$$\begin{pmatrix} AP + PA^T & B & (CP)^T \\ B^T & -\gamma I & D^T \\ CP & D & -\gamma I \end{pmatrix} < 0, \quad P > 0.$$

Lemma 3 [19]: Given a square matrix A , all the eigenvalues of A lie in the conic sector as shown in Fig. 3(a) if and only if there exists a matrix $X = X^T$ such that the following LMIs are satisfied,

$$\begin{pmatrix} \sin(\theta)(AX + XA^T) & \cos(\theta)(AX - XA^T) \\ \cos(\theta)(XA^T - AX) & \sin(\theta)(AX + XA^T) \end{pmatrix} < 0, \\ X > 0.$$

Lemma 4 [19]: Given a square matrix A , all the eigenvalues of A lie in the vertical strip $(-h_1, -h_2)$ as shown in Fig. 3(b) if and only if there exists a matrix $X = X^T$ such that the following LMIs are satisfied,

$$\begin{aligned} -2h_1X - AX - XA^T &< 0, \\ 2h_2X + AX + XA^T &< 0, \\ X &> 0. \end{aligned}$$

Fig.3 (a) conic sector (b) vertical strip $(-h_1, -h_2)$

Note that for standard second-order systems and their approximations, placing the dominant poles in the conic sector region (see Fig. 3(a)) with smaller θ results in smaller percent overshoot [16]. Similarly, placing the dominant poles in the vertical strip (see Fig. 3(b)) with larger h_2 results in smaller settling time [16].

III. GENERAL FORM LOOP FILTER DESIGN

In this section, a new method is proposed to design a loop filter for PLL. Firstly in part A, we generalize the technique in [23], [24] to transform the problem of designing a dynamic filter into a static state feedback synthesis problem. Secondly, in part B, a multi-objective state feedback synthesis technique is employed to find a loop filter to satisfy the design objectives mentioned in Section II.

A. Problem reformulation

To proceed, Fig. 2 is redrawn as Fig. 4 below

Fig. 4 PLL Linear model II

with loop filter $F(s)$ of the form

$$F(s) = \frac{s}{s+a} \cdot \frac{1 + (T/2) \cdot s}{(1 - f_{m+l+1}) + (T/2) \cdot s} \cdot \left[\sum_{i=0}^m \frac{f_i}{s^i} + \sum_{i=m+1}^{m+l} \frac{f_i}{s - \beta_{i-m}} \right] \quad (3)$$

where a and $\beta_{j,s}$ are the real poles of the filter which can be assigned a priori, T denotes pure delay, and the real coefficients $f_{i,s}$ are to be designed. Let $l=0$ denote the case that the component functions with poles at $\beta_{j,s}$ are taken out from (3), i.e.,

$$F(s) = \frac{s}{s+a} \cdot \frac{1 + \frac{T}{2}s}{(1 - f_{m+l+1}) + \frac{T}{2}s} \cdot \left[\sum_{i=0}^m \frac{f_i}{s^i} \right]$$

In view of Fig. 4, the signal $Y(s)$ can be described by

$$Y(s) = e^{-sT} \frac{A_d}{s+a} (W_n(s) + W_\theta(s) - \hat{U}(s)) \quad (4)$$

Applying first-order Padé approximation for the delay term e^{-sT} by

$$e^{-sT} \approx \frac{1 - T/2 \cdot s}{1 + T/2 \cdot s} =: D(s)$$

Then (4) becomes

$$\begin{aligned} sY(s) &\approx -aY(s) + A_d \hat{U}(s) - 2A_d \frac{1}{1 + T/2 \cdot s} \hat{U}(s) \\ &+ [A_d \quad A_d] \bar{D}(s) \begin{bmatrix} W_n(s) \\ W_\theta(s) \end{bmatrix} \end{aligned} \quad (5)$$

where $\bar{D}(s) = \text{diag}(D(s), D(s))$ is an all-pass transfer matrix.

Define state vector $\bar{\xi}(s)$ in frequency domain as follows:

$$\begin{aligned} \bar{\xi}(s) &:= \left[Y(s), \frac{1}{s}Y(s), \dots, \frac{1}{s^m}Y(s), \frac{1}{s - \beta_1}Y(s), \dots, \frac{1}{s - \beta_l}Y(s), \frac{1}{1 + T/2 \cdot s} \hat{U}(s) \right]^T \\ &=: [\xi_1(s), \xi_2(s), \xi_3(s), \dots, \xi_{m+l+2}(s)]^T. \end{aligned}$$

Then (5) becomes

$$s\xi_1(s) = -a\xi_1(s) - 2A_d\xi_{m+l+2}(s) + A_d\hat{U}(s) + [A_d \quad A_d] \bar{D}(s) \begin{bmatrix} W_n(s) \\ W_\theta(s) \end{bmatrix}$$

Moreover, the other state equations read

$$\begin{aligned} s\xi_i(s) &= \xi_{i-1}(s), \quad i = 2, 3, \dots, m+1 \\ s\xi_i(s) &= \beta_{i-m-1} \cdot \xi_i(s) + \xi_1, \quad i = m+2, \dots, m+l+1 \\ s\xi_{m+l+2}(s) &= -2/T \cdot \xi_{m+l+2}(s) + 2/T \cdot \hat{U}(s). \end{aligned}$$

Set the observed output $Z(s) := \hat{U}(s)$ and $\bar{Y}(s) := \bar{\xi}(s)$. With the notation defined above it is easy to check that the dynamic filter of the prescribed form (3) is converted into a static state feedback in the new coordinate, i.e., $\hat{U}(s) = F_s \cdot \bar{Y}(s)$ where $F_s = [f_0, \dots, f_{m+l+1}]$. Thus the original dynamic filter design problem is equivalently transformed to a static state feedback control problem as illustrated in Fig. 5

Fig. 5 The equivalent state feedback model

with G described by

$$G \begin{cases} s\bar{\xi}(s) = A\bar{\xi}(s) + B_1\bar{W}_d(s) + B_2\hat{U}(s) \\ Z(s) = D_{12}\hat{U}(s) \\ \bar{Y}(s) = \bar{\xi}(s) \end{cases}$$

where $\bar{W}_d(s) = [W_{d_1}(s) \ W_{d_2}(s)]^T := \bar{D}(s) \cdot [W_n(s) \ W_\theta(s)]^T$,

$$A = \begin{bmatrix} -a & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -2A_d \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots & 0 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 & \vdots & \ddots & \vdots & 0 \\ 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & \beta_1 & 0 & \cdots & 0 \\ 1 & 0 & \ddots & \vdots & \vdots & 0 & \beta_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & 0 & \vdots & \ddots & \vdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \beta_l \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -2/T \end{bmatrix}_{(m+l+2) \times (m+l+2)}$$

$$B_1 = \begin{bmatrix} A_d & 0 & \cdots & 0 \\ A_d & 0 & \cdots & 0 \end{bmatrix}_{2 \times (m+l+2)}^T, \quad B_2 = [A_d \ 0 \ \cdots \ 0 \ 2/T]_{1 \times (m+l+2)}^T,$$

$$D_{12} = 1.$$

The resulting closed-loop system from $\bar{W}_d(s)$ to $Z(s)$ is thus as follows.

$$T_{z\bar{W}_d} \begin{cases} s\bar{\xi}(s) = (A + B_2F_s)\bar{\xi}(s) + B_1\bar{W}_d(s) \\ Z(s) = F_s\bar{\xi}(s) \end{cases}$$

Since $D(s)$ is an all-pass transfer function, for H_2 minimization from $W_n(s)$ to $Z(s)$, equivalently from $W_{d_1}(s)$ to $Z(s)$, the system matrix of the closed-loop map is given by

$$\begin{bmatrix} A + B_2F_s & B_1R_1 \\ F_s & 0 \end{bmatrix}$$

where $R_1 = [1 \ 0]^T$. On the other hand, for H_∞ minimization from $W_\theta(s)$ to $Z(s)$, equivalently from $W_{d_2}(s)$ to $Z(s)$, the system matrix of the closed-loop map is given by

$$\begin{bmatrix} A + B_2F_s & B_1R_2 \\ F_s & 0 \end{bmatrix}$$

$$\text{where } R_2 = [0 \ 1]^T.$$

B. LMI design of the general form loop filters

In order to meet the multiple design objectives mentioned in Section II, the following multi-objective control problem is considered, where λ_2 and λ_∞ are weights for the trade-off of the two design objectives: noise bandwidth and gain/phase margins.

$$\text{Minimize } \lambda_2 \cdot \nu^2 + \lambda_\infty \cdot \gamma \quad (6)$$

$$\text{over } N \in \Re^{1 \times (m+l+2)}, \ M = M^T \in \Re^{(m+l+2) \times (m+l+2)}, \ Q \in \Re,$$

$\nu \in \Re$, and $\gamma \in \Re$ satisfying:

$$\begin{cases} \begin{pmatrix} He(AM + B_2N) & B_1R_1 \\ (B_1R_1)^T & -I \end{pmatrix} < 0, \quad \begin{pmatrix} M & N^T \\ N & Q \end{pmatrix} > 0, \\ Q < \nu^2 \end{cases}$$

$$\begin{pmatrix} He(AM + B_2N) & B_1R_2 & N^T \\ (B_1R_2)^T & -\gamma I & 0 \\ N & 0 & -\gamma I \end{pmatrix} < 0$$

$$\begin{pmatrix} \sin(\theta) \cdot He(AM + B_2N) & \cos(\theta) \cdot Sh(AM + B_2N) \\ (\cos(\theta) \cdot Sh(AM + B_2N))^T & \sin(\theta) \cdot He(AM + B_2N) \end{pmatrix} < 0$$

$$\begin{cases} -2h_1M - He(AM + B_2N) < 0 \\ 2h_2M + He(AM + B_2N) < 0 \end{cases}$$

where the notation $He(A) = A + A^T$, $Sh(A) = A - A^T$ are used, $\Re$ denotes the set of all real numbers, and $\Re^{p \times m}$ denotes the set of all $p \times m$ matrices with all the entries in $\Re$. Denote the optimal solution by $(N_{opt}, M_{opt}, Q_{opt}, \nu_{opt}, \gamma_{opt})$.

Theorem 1: Given the time delay T , nonnegative integers l, m , the real values $a, \beta_1, \beta_2, \dots, \beta_l$, the weights $\lambda_2, \lambda_\infty$, the positive values θ , and h_1, h_2 with $h_2 < h_1$. If the optimization problem (6) is solvable, there exists a loop filter of the form

$$F(s) = \frac{s}{s+a} \cdot \frac{1+(T/2) \cdot s}{(1-f_{m+l+1})+(T/2) \cdot s} \cdot \left[\sum_{i=0}^m \frac{f_i}{s^i} + \sum_{i=m+1}^{m+l} \frac{f_i}{s-\beta_{l-m}} \right]$$

with $F_s = [f_0, \dots, f_{m+l+1}] = N_{opt} M_{opt}^{-1}$, such that

(i) the closed-loop system is stable,

(ii) $B_n < \frac{\nu_{opt}^2}{2}$, $\text{GM} > 20 \log \frac{\gamma_{opt} + 1}{\gamma_{opt}} \text{dB}$, $\text{PM} > 2 \sin^{-1} \frac{1}{2\gamma_{opt}} \text{deg}$,

(iii) the closed-loop poles lie within the intersection of the strip $(-h_1, -h_2)$ and the conic sector with parameter θ .

Proof: The assertions can be proved by applying Lemmas 1,2,3,4 and the standard state feedback synthesis technique from [17], and the formulas (1) and (2).

Remark: The proposed design method provides an advantage for the designer to assign the filter poles a priori except that at $2(f_{m+l+1}-1)/T$. Moreover, the method is applicable to yield a filter of arbitrary order. Besides, the tracking capability of PLL loops can be enforced by increasing the number of the pure integrators in the filter.

An algorithm based on carrying out the optimization problem (6) and adjustment of the parameters h_1, h_2 and θ is

presented as follows.

Algorithm 1: Given the desired noise bandwidth B_n^* (Hz), stability margins, i.e., GM/PM (equivalently $\|T_{\hat{w}_\theta w_\theta}\|_\infty = \gamma^*$),

both of which are assumed to be greater than the minimal values in the individual case. Consider loop filter of form (3). Without loss of generality, λ_2 may be assumed to be unity.

Step 1 Select the weight λ_∞ , the values h_1 , h_2 and θ .

Perform (6) to get the values of B_n and γ .

Step 2 If $B_n > B_n^*$, decreases h_2 and / or h_1 . If $\gamma > \gamma^*$, decreases h_2 and/or increases h_1 . Continue this process until $B_n \leq B_n^*$ and $\gamma \leq \gamma^*$.

Step 3 If the transient response is satisfactory, the filter design is complete; otherwise continue the following steps:

Step 3.1 If it is not good enough for overshoot, we decrease θ and perform (6) to get B_n and γ .

Go to Step 2.

Step 3.2 If it is not good enough for settling time, we increase h_2 and/or θ and perform (6) to get B_n and γ . Go to Step 2.

IV. PI FORM LOOP FILTER DESIGN

In this section, a new design of PI form loop filters is proposed for PLLs. The synthesis problem is formulated as a mixed H_2/H_∞ control problem. A special coordinate transformation technique [21] is employed to deal with the problem.

A. Problem reformulation

In view of the formula (3) for the loop filter, if $l=0$ and both of the coefficients a and f_{m+1} are set to be zero, then (3) reduces to a PI form filter of order m . Specifically, define the state vector as

$$X(s) := \left[Y(s), \frac{1}{s}Y(s), \dots, \frac{1}{s^m}Y(s), \frac{1}{1+T/2 \cdot s} \hat{U}(s) \right]^T$$

With similar argument as part A of Section III, it is easy to verify that the resulting closed-loop system from $\bar{W}_d(s)$ to $Z(s)$ is as follows.

$$T_{z\bar{w}_d} \begin{cases} sX(s) = (A + B_2 F_0 C_2)X(s) + B_1 \bar{W}_d(s) \\ Z(s) = F_0 C_2 X(s) \end{cases}$$

where

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & -2A_d \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & -2/T \end{bmatrix}_{(m+2) \times (m+2)},$$

$$B_1 = \begin{bmatrix} A_d & 0 & \cdots & 0 \\ A_d & 0 & \cdots & 0 \end{bmatrix}_{2 \times (m+2)}^T, \quad B_2 = \begin{bmatrix} A_d & 0 & \cdots & 0 & 2/T \end{bmatrix}_{1 \times (m+2)}^T,$$

$$C_2 = \begin{bmatrix} I_{m+1} & 0_{(m+1) \times 1} \end{bmatrix}, \text{ and } F_0 = [f_0, \dots, f_m] \text{ is to be designed.}$$

Again, for H_2 minimization from $W_n(s)$ to $Z(s)$ and for H_∞ minimization from $W_\theta(s)$ to $Z(s)$ using m th order PI form filter, it is required to solve (7) and (8) below, respectively, for the variables Q_2 , V , Q_∞ and F_0 .

$$\begin{cases} \begin{pmatrix} He\{AQ_2 + B_2 F_0 C_2 Q_2\} & B_1 R_1 \\ (B_1 R_1)^T & -1 \end{pmatrix} < 0, \\ \begin{pmatrix} Q_2 & (F_0 C_2 Q_2)^T \\ F_0 C_2 Q_2 & V \end{pmatrix} > 0, \\ Tr(V) < \nu^2. \end{cases} \quad (7)$$

$$\begin{cases} \begin{pmatrix} He\{AQ_\infty + B_2 F_0 C_2 Q_\infty\} & B_1 R_2 & (F_0 C_2 Q_\infty)^T \\ (B_1 R_2)^T & -\gamma I & 0 \\ F_0 C_2 Q_\infty & 0 & -\gamma I \end{pmatrix} < 0, \\ Q_\infty > 0 \end{cases} \quad (8)$$

Unfortunately, neither (7) nor (8) is jointly convex in the variables of its own. In the next subsection, we introduce a newly developed coordinate transformation technique [21] which will be used to deal with this difficult problem.

B. Coordinate transformation technique

Given $A \in \Re^{(m+2) \times (m+2)}$, $B_1 \in \Re^{(m+2) \times 2}$, $B_2 \in \Re^{(m+2) \times 1}$ and $C_2 = \begin{bmatrix} I_{m+1} & 0_{(m+1) \times 1} \end{bmatrix}$. Define a set $\mathcal{S}$ of real matrices as follows:

$$\mathcal{S} := \left\{ \begin{pmatrix} I_{m+1} & 0_{(m+1) \times 1} \\ T_{21} & T_{22} \end{pmatrix} \in \Re^{(m+2) \times (m+2)} : T_{21} \in \Re^{1 \times (m+1)}, T_{22} \neq 0 \right\}.$$

It is readily checked that any matrix $T \in \mathcal{S}$ satisfies the property: $C_2 T = C_2$. We will make use of the following two lemmas to transform the H_2 and H_∞ minimization problems mentioned above to be LMI problems in certain coordinates, respectively.

Lemma 5 (H_2 performance) [21]: Given $\nu > 0$. The following statements are equivalent.

(i) there exist matrices $Q_2 = Q_2^T \in \Re^{(m+2) \times (m+2)}$, $V \in \Re$ and $F_0 \in \Re^{1 \times (m+1)}$ such that (7) is satisfied.

(ii) there exist matrices $N \in \Re^{1 \times (m+1)}$, $M \in \Re^{(m+1) \times (m+1)}$, $U_2 \in \Re$,

$V \in \mathfrak{R}$ and $T_2 \in \mathcal{S}$ such that

$$\begin{pmatrix} He\left\{\begin{bmatrix} \bar{\mathbf{A}} & 0 \\ 0 & U_2 \end{bmatrix} + \bar{\mathbf{B}}_2 N \bar{\mathbf{C}}_2\right\} & \bar{\mathbf{B}}_1 \\ (\bar{\mathbf{B}}_1)^T & -I \end{pmatrix} < 0, \quad (9)$$

$$\begin{pmatrix} \begin{bmatrix} M & 0 \\ 0 & U_2 \end{bmatrix} & (N \bar{\mathbf{C}}_2)^T \\ N \bar{\mathbf{C}}_2 & V \end{pmatrix} > 0, \quad Tr(V) < v^2.$$

where

$$\bar{\mathbf{A}} = T_2^{-1} A T_2, \quad \bar{\mathbf{B}}_1 = T_2^{-1} B_1 R_1, \quad \bar{\mathbf{B}}_2 = T_2^{-1} B_2, \quad \bar{\mathbf{C}}_2 = C_2 T_2.$$

Lemma 6 (H_∞ performance) [21]: Given $\gamma > 0$. The following statements are equivalent.

- (i) there exist matrices $Q_\infty = Q_\infty^T \in \mathfrak{R}^{(m+2) \times (m+2)}$ and $F_0 \in \mathfrak{R}^{1 \times (m+1)}$ such that (8) is satisfied.
- (ii) there exist matrices $N \in \mathfrak{R}^{1 \times (m+1)}$, $M \in \mathfrak{R}^{(m+1) \times (m+1)}$, $U_\infty \in \mathfrak{R}$ and $T_\infty \in \mathcal{S}$ such that

$$\begin{pmatrix} He\left\{\begin{bmatrix} \hat{\mathbf{A}} & 0 \\ 0 & U_\infty \end{bmatrix} + \hat{\mathbf{B}}_2 N \hat{\mathbf{C}}_2\right\} & \hat{\mathbf{B}}_1 & (N \hat{\mathbf{C}}_2)^T \\ (\hat{\mathbf{B}}_1)^T & -\gamma I & 0 \\ N \hat{\mathbf{C}}_2 & 0 & -\gamma I \end{pmatrix} < 0, \quad \begin{bmatrix} M & 0 \\ 0 & U_\infty \end{bmatrix} > 0, \quad (10)$$

where

$$\hat{\mathbf{A}} = T_\infty^{-1} A T_\infty, \quad \hat{\mathbf{B}}_1 = T_\infty^{-1} B_1 R_2, \quad \hat{\mathbf{B}}_2 = T_\infty^{-1} B_2, \quad \hat{\mathbf{C}}_2 = C_2 T_\infty.$$

Note that the coordinate transformation matrices T_i , $i = 2, \infty$, mentioned in statement (ii) of Lemma 5 and Lemma 6 can be set as follows.

$$T_i = \begin{pmatrix} I_{m+1} & 0 \\ ((Q_{i,11})^{-1} Q_{i,12})^T & 1 \end{pmatrix} \quad (11)$$

where $Q_i = \begin{bmatrix} Q_{i,11} & Q_{i,12} \\ Q_{i,12}^T & Q_{i,22} \end{bmatrix}$, $i = 2, \infty$, were obtained from solving (7) and (8), respectively.

C. LMI design of the PI form loop filters

The problem of PI form loop filter design is formulated as a mixed H_2/H_∞ optimization problem described as follows:

$$\text{Minimize } \lambda_2 \cdot v^2 + \lambda_\infty \cdot \gamma$$

over $F_0 \in \mathfrak{R}^{1 \times (m+1)}$, $Q_2 = Q_2^T \in \mathfrak{R}^{(m+2) \times (m+2)}$, $Q_\infty = Q_\infty^T \in \mathfrak{R}^{(m+2) \times (m+2)}$, $V \in \mathfrak{R}$, $v \in \mathfrak{R}$, and $\gamma \in \mathfrak{R}$ satisfying (7) and (8)

Applying the coordinate transformation technique equivalently transforms the mixed H_2/H_∞ optimization problem into the following optimization problem:

$$\text{Minimize } \lambda_2 \cdot v^2 + \lambda_\infty \cdot \gamma \quad (12)$$

over $N \in \mathfrak{R}^{1 \times (m+1)}$, $M \in \mathfrak{R}^{(m+1) \times (m+1)}$, $U_2 \in \mathfrak{R}$, $U_\infty \in \mathfrak{R}$, $V \in \mathfrak{R}$, $v \in \mathfrak{R}$, and $\gamma \in \mathfrak{R}$ satisfying: (9) and (10).

Denote the locally optimal solution by

$$(\tilde{U}_2, \tilde{U}_\infty, \tilde{N}, \tilde{M}, \tilde{V}, \tilde{v}, \tilde{\gamma})$$

Theorem 2: Suppose $m > 0$, and the weights λ_2 and λ_∞ are given. If the optimization problem (12) is solvable, there exists a PI form loop filter $F(s)$ expressed as

$$F(s) = \sum_{i=0}^m \frac{f_i}{s^i}$$

with $F_0 = [f_0, \dots, f_m] = \tilde{N} \tilde{M}^{-1}$, such that

(i) the closed-loop system is stable,

(ii) $B_n < \frac{\tilde{v}^2}{2}$, $\text{GM} > 20 \log \frac{\tilde{\gamma}+1}{\tilde{\gamma}} \text{ dB}$, $\text{PM} > 2 \sin^{-1} \frac{1}{2\tilde{\gamma}} \text{ deg}$.

Proof: The assertions can be proved by applying Lemmas 5 and 6 and the formulas (1) and (2).

An algorithm based on carrying out the optimization problem (12) is presented as follows.

Algorithm 2: Given the desired noise bandwidth B_n^* (Hz), stability margins (equivalently, $\|T_{w_0 w_p}\|_\infty = \gamma^*$) and the order m of the loop filter. The weight λ_2 is selected to unity.

Step 1 Select initial feedback gain F_0 .

Step 2 Solve (7) for minimizing v to get Q_2 , and solve (8) for minimizing γ to get Q_∞ .

Step 3 Compute T_i , $i = 2, \infty$, by (11).

Step 4 Compute $\bar{\mathbf{A}}$, $\bar{\mathbf{B}}_1$, $\bar{\mathbf{B}}_2$, $\bar{\mathbf{C}}_2$, $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}_1$, $\hat{\mathbf{B}}_2$, $\hat{\mathbf{C}}_2$.

Step 5 Given λ_∞ . Perform (12) to get the values B_n and γ . If

$B_n > B_n^*$, decreases λ_∞ . If $\gamma > \gamma^*$ increases λ_∞ .

Continue this process until $B_n \leq B_n^*$ and $\gamma \leq \gamma^*$.

V. SIMULATION RESULTS

In practical situation, an appropriate noise bandwidth of PLL in GPS receiver is selected to be around 15 Hz [25], i.e. $B_n^* \approx 15$ Hz. This design objective is enforced for all the cases discussed here. Without loss of generality, the phase detector gain A_d is assumed to be unity. Furthermore, we assume that the loop delay is 10 ms and the standard deviation of the input Gaussian noise $n(t)$ is 0.05. Two kinds of loop filter are designed by the proposed methods. The general form filters considered here are expressed by:

$$F(s) = \frac{s + \frac{2}{T}}{s + (1 - f_{m+1}) \cdot \frac{2}{T}} \cdot \sum_{i=0}^m \frac{f_i}{s^i}.$$

Note that the class of PI form filters can be deemed as a special case of the general form filters because a general form filter reduces to be PI form when the parameter f_{m+1} is set to be zero. The two types of filters are computed via Algorithm 1 and Algorithm 2.

TABLE 1 (A) and (B) show the transfer functions of the two types of loop filters and their design parameters. In TABLE 1 (B), the value inside each bracket denotes the number of iteration. For example, the symbol $F_0(0)$ denotes the feedback gain initially chosen. Several performance indices are evaluated for the resulting PLLs as shown in TABLE 2. The gain and phase margins on the list are the guaranteed values computed by the formulas (1) and (2), which are approximately the same for both designs. For overshoot performance index, the resulting PLLs yield almost the same quantity. Compared with PI form filters, the general form filters have slightly smaller settling time.

TABLE 1 (A) GENERAL FORM LOOP FILTERS

# integrator	Design parameters	Loop filter $F(s)$
1	$\lambda_2 = 1, \lambda_{\infty} = 1$ $\theta = \pi/11.64, h_1 = 210, h_2 = 12$	$\frac{(46.3s + 392.4) \cdot (s + 200)}{s(s + 288.68)}$
2	$\lambda_2 = 1, \lambda_{\infty} = 15$ $\theta = \pi/40, h_1 = 190, h_2 = 3.48$	$\frac{(46.85s^2 + 326.18s + 628.16) \cdot (s + 200)}{s^2(s + 282.44)}$
3	$\lambda_2 = 1, \lambda_{\infty} = 20$ $\theta = \pi/55, h_1 = 150, h_2 = 2.03$	$\frac{(38.5s^3 + 303.95s^2 + 828.97s + 733.66) \cdot (s + 200)}{s^3(s + 235.08)}$

TABLE 1 (B) PI FORM LOOP FILTERS

# integrator	Design parameters	Loop filter $F(s)$
1	$F_0(0) = [50 \ 510]$, $\lambda_2(0) = 1, \lambda_{\infty}(0) = 362$, $F_0(1) = [33.36 \ 267.94]$	$33.36 + \frac{267.94}{s}$
2	$F_0(0) = [100 \ 1000 \ 3000]$, $\lambda_2(0) = 1, \lambda_{\infty}(0) = 330$, $F_0(1) = [34.44 \ 226.19 \ 497.08]$	$34.44 + \frac{226.19}{s} + \frac{497.08}{s^2}$
3	$F_0(0) = [100 \ 1000 \ 1970 \ 2970]$, $\lambda_2(0) = 1, \lambda_{\infty}(0) = 284$, $F_0(1) = [33.20 \ 266.16 \ 517.88 \ 742.60]$	$33.2 + \frac{266.16}{s} + \frac{517.88}{s^2} + \frac{742.6}{s^3}$

TABLE 2 The performance indices for noise bandwidth 15 Hz

Filter type	# integrator	Rise time (sec)	Overshoot (%)	Settling time (sec)	H_{∞} norm
General form	1	0.03	17.8	0.306	1.21
	2	0.03	16.8	0.379	1.2
	3	0.03	20	0.355	1.24
PI Form	1	0.028	18.1	0.311	1.21
	2	0.028	16.8	0.388	1.19
	3	0.029	20	0.362	1.24

Filter type	# integrator	Gain margin	Phase margin
General form	1	5.23 dB	48.81 deg
	2	5.26 dB	49.25 deg
	3	5.14 dB	47.56 deg
PI Form	1	5.23 dB	48.81 deg
	2	5.26 dB	49.69 deg
	3	5.14 dB	47.56 deg

Next we use the software MATLAB's SIMULINK Version 7.3 to simulate the resulting PLLs. A nonlinear system model, as shown in Fig. 6, is used, where the function of the phase detector is modeled as $A_d \sin(\bullet)$ instead of a constant gain A_d .

Figures 7, 8 and 9 show the frequency step responses of second-order, third-order and fourth-order PLLs,

respectively. Overall, these show that the resulting PLLs using both general form filters and PI form filters can track the input trajectory swiftly when the phase ramp signal of 40 rad/sec is applied. The transient responses of the resulting PLLs are similar.

Fig. 6 The block diagram of PLL model

Fig. 7 The frequencies of VCO output when a frequency change (ramp input phase): (a) 40 rad/sec and (b) 75 rad/sec is applied to second-order PLLs.

Fig. 8 The frequencies of VCO output when a frequency change (ramp input phase): (a) 40 rad/sec and (b) 75 rad/sec is applied to third-order PLLs.

The pull-out phenomenon occurs if a larger frequency offset signal of 75 rad/sec is encountered. Through a slow pull-in process, the loop can get locked again as long as it is within the lock-in range. It is observed that the PLLs with

general form loop filters exhibit better tracking capability and shorter pull-in time than those with PI form filters. Particularly, when the PLLs suffer from a greater frequency offset, those with general form filters can provide an effective control to speed up the tracking capability. Nevertheless, the performances of the PI form filters are very close to those of the general form filters and are easier to be implemented.

Fig. 9 The frequencies of VCO output when a frequency change (ramp input phase): (a) 40 rad/sec and (b) 75 rad/sec is applied to fourth-order PLLs.

VI. CONCLUSIONS

We have presented two new loop filter designs for PLLs with undesired but inevitable loop delay. Both of the two methods considered to tradeoff the design objectives such as small noise bandwidth and large gain/phase margins via multi-objective control techniques. When first-order Padé approximation for the delay term is applied, unlike the results of direct application of the existing multi-objective control technique leading to filters of only second-order which may not be able to meet the increasingly tough performance requirements of PLLs, our proposed methods can yield filters with arbitrary order. Furthermore, almost all except one of the general form filter poles can be assigned a priori. A special coordinate transformation technique has been employed to deal with the design of the popular class of PI form filters. Algorithms have been provided for the proposed design methods. Simulation results demonstrate the good performance of the resulting PLLs.

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