TIME HARMONIC ELECTROMAGNETIC FIELDS IN AN BIAxIAL ANISOTROPIC MEDIUM

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Abstract—This study develops the scalar wave equations of auxiliary vector potentials for time harmonic electric and magnetic fields on transverse electric, transverse magnetic and transverse electromagnetic modes in an biaxial anisotropic medium. The concise scalar wave equations are achieved in both rectangular and cylindrical coordinates. Moreover, in an biaxial medium, it is identified that the decoupled TE, TM, and TEM modes can only exist under certain conditions for various modes respectively which clarify the suspect about the existence of these modes that was reported previously.

1 Introduction

2 Derivation of Scalar Wave Equations

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1. INTRODUCTION

The analysis of electromagnetic field for anisotropic object is an important issue. Work has been conducted on various areas [1–14]. It is recognized that the widely used substrate material sapphire is anisotropic. The lithium niobate (LiNbO_3) is used in the design of integrated optics devices and has an uniaxial permittivity tensor. Therefore, analyses for devices based on these materials must account for the possibility of anisotropic material tensor.

For the electromagnetic boundary, the most widely known field configurations are those referred to as the transverse electric (TE), transverse magnetic (TM), and transverse electromagnetic (TEM) modes. Work on the electromagnetic fields of these modes for an biaxial anisotropic medium remains incomplete. Moreover, it was reported [3, 4] that the biaxial case does not allow the decomposition of field into TE and TM modes.

A different aspect was reached in this work. The TE, TM, and TEM modes do exist but only under especial conditions. The completed expressions for these modes will be given. The required condition for each mode to exist will be presented.

The auxiliary vector potentials are well known to be F and A , where

$$D = -\nabla \times F \quad (1)$$

$$B = \nabla \times A \quad (2)$$

can be used to solve the electric and magnetic fields of the TE, TM, and TEM modes in an isotropic medium, and can be found in some electromagnetics books [15]. This study develops the scalar wave equations of A or F for TE, TM, and TEM modes transverse to various directions ($TE^{x,y,z}$, $TM^{x,y,z}$, and $TEM^{x,y,z}$) in both rectangular and cylindrical coordinates in an biaxial anisotropic medium. The electric and magnetic fields in an anisotropic medium with various geometries can then be obtained by solving the scalar wave equations with adequate boundary conditions. The derivation procedures are shown to be quite simple. Since the individual procedures for different modes are similar, this study only presents detailed procedures for the first derived TE^y mode.

2. DERIVATION OF SCALAR WAVE EQUATIONS

The work is focused on the biaxial anisotropic medium, where the permittivity and permeability are

$$\varepsilon = \begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} \quad \text{and} \quad \mu = \begin{bmatrix} \mu_x & 0 & 0 \\ 0 & \mu_y & 0 \\ 0 & 0 & \mu_z \end{bmatrix} \quad (3)$$

respectively. Therefore,

$$D = \hat{a}_x D_x + \hat{a}_y D_y + \hat{a}_z D_z = \hat{a}_x \varepsilon_x E_x + \hat{a}_y \varepsilon_y E_y + \hat{a}_z \varepsilon_z E_z \quad (4)$$

$$B = \hat{a}_x B_x + \hat{a}_y B_y + \hat{a}_z B_z = \hat{a}_x \mu_x H_x + \hat{a}_y \mu_y H_y + \hat{a}_z \mu_z H_z \quad (5)$$

for rectangular coordinate, and

$$D = \hat{a}_r D_r + \hat{a}_\phi D_\phi + \hat{a}_z D_z = \hat{a}_r \varepsilon_r E_r + \hat{a}_\phi \varepsilon_\phi E_\phi + \hat{a}_z \varepsilon_z E_z \quad (6)$$

$$B = \hat{a}_r B_r + \hat{a}_\phi B_\phi + \hat{a}_z B_z = \hat{a}_r \mu_r H_r + \hat{a}_\phi \mu_\phi H_\phi + \hat{a}_z \mu_z H_z \quad (7)$$

for cylindrical coordinate, where

$$\frac{(\varepsilon, \mu)_r}{(\varepsilon, \mu)_x} \cos^2 \phi + \frac{(\varepsilon, \mu)_r}{(\varepsilon, \mu)_y} \sin^2 \phi = 1 \quad (8)$$

$$\frac{(\varepsilon, \mu)_\phi}{(\varepsilon, \mu)_x} \sin^2 \phi + \frac{(\varepsilon, \mu)_\phi}{(\varepsilon, \mu)_y} \cos^2 \phi = 1 \quad (9)$$

Source free and time harmonic will be assumed in the following derivations.

(A) TE Modes

The rectangular coordinate is firstly considered and starts with the TE to the y direction (TE^y mode). For TE^y mode, let

$$F = \hat{a}_y F_y(x, y, z); \quad F_x = F_z = 0 \quad (10)$$

From equation (1),

$$E_x = \frac{1}{\varepsilon_x} \frac{\partial F_y}{\partial z}; \quad E_y = 0; \quad E_z = -\frac{1}{\varepsilon_z} \frac{\partial F_y}{\partial x} \quad (11)$$

From equation (4) and Maxwell's equation

$$\nabla \times E = -j\omega B, \quad (12)$$

we can get,

$$\frac{\partial E_z}{\partial y} = -j\omega \mu_x H_x; \quad \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -j\omega \mu_y H_y; \quad \frac{\partial E_x}{\partial y} = j\omega \mu_z H_z \quad (13)$$

In addition, from equation (1),

$$\nabla \times D = \nabla^2 F - \nabla \nabla \cdot F \quad (14)$$

and by adding the following equation,

$$H = -\nabla \phi_m - j\omega F \quad (15)$$

where ϕ_m represents a magnetic scalar potential. We have

$$\begin{aligned}
 \nabla \times D &= \hat{a}_x \varepsilon_z \frac{\partial E_z}{\partial y} + \hat{a}_y \left(\varepsilon_x \frac{\partial E_x}{\partial z} - \varepsilon_z \frac{\partial E_z}{\partial x} \right) - \hat{a}_z \varepsilon_x \frac{\partial E_x}{\partial y} \\
 &= -\hat{a}_x j\omega \mu_x \varepsilon_z H_x + \hat{a}_y \left[-j\omega \mu_y \varepsilon_x H_y + (\varepsilon_x - \varepsilon_z) \frac{\partial E_z}{\partial x} \right] - \hat{a}_z j\omega \mu_z \varepsilon_x H_z \\
 &= \hat{a}_x j\omega \mu_x \varepsilon_z \frac{\partial \phi_m}{\partial x} + \hat{a}_y \left[j\omega \mu_y \varepsilon_x \frac{\partial \phi_m}{\partial y} - \omega^2 \mu_y \varepsilon_x F_y + \left(1 - \frac{\varepsilon_x}{\varepsilon_z} \right) \frac{\partial^2 F_y}{\partial x^2} \right] \\
 &\quad + \hat{a}_z j\omega \mu_z \varepsilon_x \frac{\partial \phi_m}{\partial z}
 \end{aligned} \tag{16}$$

and

$$\nabla^2 F - \nabla \nabla \cdot F = -\hat{a}_x \frac{\partial^2 F_y}{\partial x \partial y} + \hat{a}_y \left(\frac{\partial^2 F_y}{\partial x^2} + \frac{\partial^2 F_y}{\partial z^2} \right) - \hat{a}_z \frac{\partial^2 F_y}{\partial z \partial y} \tag{17}$$

By equating the equations (16) and (17) on x and z components, we can have

$$\frac{\partial^2 F_y}{\partial x \partial y} = -j\omega \mu_x \varepsilon_z \frac{\partial \phi_m}{\partial x} \tag{18}$$

$$\frac{\partial^2 F_y}{\partial z \partial y} = -j\omega \mu_z \varepsilon_x \frac{\partial \phi_m}{\partial z} \tag{19}$$

By solving the above two equations, we can have

$$\frac{\partial F_y}{\partial y} = -j\omega \mu_x \varepsilon_z \phi_m + C \tag{20}$$

$$\frac{\mu_x}{\varepsilon_x} = \frac{\mu_z}{\varepsilon_z} \tag{21}$$

The equation (21) can be seen as a condition required for the TE^y mode to exist. Finally, by equating the y components of equations of (16) and (17) and using equation (21), we can have the scalar wave equation for vector potential component F_y ,

$$\mu_x \frac{\partial^2 F_y}{\partial x^2} + \mu_y \frac{\partial^2 F_y}{\partial y^2} + \mu_z \frac{\partial^2 F_y}{\partial z^2} = -\omega^2 \mu_y \varepsilon_x \mu_z F_y = -\omega^2 \mu_y \varepsilon_z \mu_x F_y \tag{22}$$

A concise scalar wave equation for F_y is reached and can be used to solve the electromagnetic fields of TE^y modes with adequate boundary conditions. From equations (10), (11), (13), (21), and (22), we can summarize the results for TE^y mode:

$$\begin{aligned}
F &= \hat{a}_y F_y(x, y, z); \\
\mu_x \frac{\partial^2 F_y}{\partial x^2} + \mu_y \frac{\partial^2 F_y}{\partial y^2} + \mu_z \frac{\partial^2 F_y}{\partial z^2} + \kappa^2 F_y &= 0, \quad \kappa^2 = \omega^2 \mu_y \varepsilon_x \mu_z = \omega^2 \mu_y \varepsilon_z \mu_x; \\
\frac{\mu_x}{\varepsilon_x} &= \frac{\mu_z}{\varepsilon_z}; \\
E_x &= \frac{1}{\varepsilon_x} \frac{\partial F_y}{\partial z}, \quad E_y = 0, \quad E_z = -\frac{1}{\varepsilon_z} \frac{\partial F_y}{\partial x}; \\
H_x &= -j \frac{1}{\omega \mu_x \varepsilon_z} \frac{\partial^2 F_y}{\partial x \partial y}, \quad H_y = j \frac{1}{\omega \mu_y \varepsilon_x} \frac{\partial^2 F_y}{\partial z^2} + j \frac{1}{\omega \mu_y \varepsilon_z} \frac{\partial^2 F_y}{\partial x^2}, \\
H_z &= -j \frac{1}{\omega \mu_z \varepsilon_x} \frac{\partial^2 F_y}{\partial z \partial y}
\end{aligned}$$

For TE^x and TE^z modes, let $F = \hat{a}_x F_x(x, y, z)$ and $F = \hat{a}_z F_z(x, y, z)$ respectively. With the similar procedures as those of TE^y mode from equations (10) to (22), we can obtain the results for TE^x and TE^z modes and those equations are put in the appendix for reference.

Next consider the cylindrical coordinate. For the TE to z direction (TE^z mode), with the similar procedures as those of the rectangular coordinate, the results are summarized as following:

$$\begin{aligned}
F &= \hat{a}_z F_z(r, \phi, z); \\
\mu_r \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial F_z}{\partial r} \right) + \mu_\phi \frac{1}{r^2} \frac{\partial^2 F_z}{\partial y^2} + \mu_z \frac{\partial^2 F_z}{\partial z^2} + \kappa^2 F_z &= 0; \\
\kappa^2 &= \omega^2 \mu_z \varepsilon_\phi \mu_r = \omega^2 \mu_z \varepsilon_r \mu_\phi; \\
\frac{\mu_r}{\varepsilon_r} &= \frac{\mu_\phi}{\varepsilon_\phi}; \\
E_r &= -\frac{1}{\varepsilon_r} \frac{1}{r} \frac{\partial F_z}{\partial \phi}, \quad E_\phi = \frac{1}{\varepsilon_\phi} \frac{\partial F_z}{\partial r}, \quad E_z = 0; \\
H_r &= -j \frac{1}{\omega \mu_r \varepsilon_\phi} \frac{\partial^2 F_z}{\partial r \partial z}, \quad H_\phi = -j \frac{1}{\omega \mu_\phi \varepsilon_r} \frac{1}{r} \frac{\partial^2 F_z}{\partial \phi \partial z}, \\
H_z &= j \frac{1}{\omega \mu_z \varepsilon_\phi} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial F_z}{\partial r} \right) + j \frac{1}{\omega \mu_z} \frac{1}{r^2} \frac{\partial}{\partial \phi} \left(\frac{1}{\varepsilon_r} \frac{\partial F_z}{\partial \phi} \right).
\end{aligned}$$

In the above equations of the TE^z mode in the cylindrical coordinate, the relation of permittivity and permeability shows,

$$\frac{\mu_r}{\varepsilon_r} = \frac{\mu_\phi}{\varepsilon_\phi} \quad (23)$$

By using equations (8) and (9), this equation can be simplified to,

$$\frac{\mu_x}{\varepsilon_x} = \frac{\mu_y}{\varepsilon_y} \quad (24)$$

which agrees with the result of the TE^z mode in the rectangular coordinate. In addition, there is an extra $\mu_r[(\varepsilon_r - \varepsilon_\phi) \frac{\partial}{\partial \phi} \frac{1}{\varepsilon_r} + \frac{1}{\varepsilon_r} \frac{\partial \varepsilon_r}{\partial \phi}] \frac{1}{r^2} \frac{\partial F_z}{\partial \phi}$ in the scalar wave equation which comes from the fact that the ε_r and ε_ϕ are ϕ dependent. This extra term can be eliminated if ε_r and ε_ϕ are ϕ independent or $\varepsilon_x = \varepsilon_y$ (uniaxial).

(B) TM Modes

With the similar roles of D , E , F , and ϕ_m for the TE modes, the relations among B , H , A , and ϕ_e can be used to derive the scalar wave equations of A_x , A_y , and A_z for TM^x , TM^y , and TM^z modes respectively in a biaxial anisotropic medium. By using equations (2), (5), Maxwell's equation

$$\nabla \times H = j\omega D \quad (25)$$

and

$$E = -\nabla \phi_e - j\omega A \quad (26)$$

we can have the results for TM modes by similar procedures as from equations (10) to (22). In the rectangular coordinate for TM^y mode:

$$\begin{aligned} A &= \hat{a}_y A_y(x, y, z); \\ \varepsilon_x \frac{\partial^2 A_y}{\partial x^2} + \varepsilon_y \frac{\partial^2 A_y}{\partial y^2} + \varepsilon_z \frac{\partial^2 A_y}{\partial z^2} + \kappa^2 A_y &= 0, \quad \kappa^2 = \omega^2 \varepsilon_y \varepsilon_x \mu_z = \omega^2 \varepsilon_y \varepsilon_z \mu_x; \\ \frac{\mu_x}{\varepsilon_x} &= \frac{\mu_z}{\varepsilon_z}; \\ H_x &= -\frac{1}{\mu_x} \frac{\partial A_y}{\partial z}, \quad H_y = 0, \quad H_z = \frac{1}{\mu_z} \frac{\partial A_y}{\partial x}; \\ E_x &= -j \frac{1}{\omega \varepsilon_x \mu_z} \frac{\partial^2 A_y}{\partial x \partial y}, \quad E_y = j \frac{1}{\omega \varepsilon_y \mu_x} \frac{\partial^2 A_y}{\partial z^2} + j \frac{1}{\omega \varepsilon_y \mu_z} \frac{\partial^2 A_y}{\partial x^2}, \\ E_z &= -j \frac{1}{\omega \varepsilon_z \mu_x} \frac{\partial^2 A_y}{\partial z \partial y} \end{aligned}$$

The results for TM^x and TM^z modes are listed in appendix for reference.

For the cylindrical coordinate in TM^z mode:

$$\begin{aligned}
 A &= \hat{a}_z A_z(r, \phi, z); \\
 \varepsilon_r \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial A_z}{\partial r} \right) + \varepsilon_\phi \frac{1}{r^2} \frac{\partial^2 A_z}{\partial y^2} + \varepsilon_z \frac{\partial^2 A_z}{\partial z^2} + \kappa^2 A_z \\
 &= \varepsilon_r \left[(\mu_r - \mu_\phi) \frac{\partial}{\partial \phi} \frac{1}{\mu_r} + \frac{1}{\mu_r} \frac{\partial \mu_r}{\partial \phi} \right] \frac{1}{r^2} \frac{\partial A_z}{\partial \phi}, \\
 \kappa^2 &= \omega^2 \varepsilon_z \varepsilon_\phi \mu_r = \omega^2 \varepsilon_z \varepsilon_r \mu_\phi; \\
 \frac{\mu_r}{\varepsilon_r} &= \frac{\mu_\phi}{\varepsilon_\phi}; \\
 H_r &= \frac{1}{\mu_r} \frac{1}{r} \frac{\partial A_z}{\partial \phi}, \quad H_\phi = -\frac{1}{\mu_\phi} \frac{\partial A_z}{\partial r}, \quad H_z = 0; \\
 E_r &= -j \frac{1}{\omega \varepsilon_r \mu_\phi} \frac{\partial^2 A_z}{\partial r \partial z}, \quad E_\phi = -j \frac{1}{\omega \varepsilon_\phi \mu_r} \frac{1}{r} \frac{\partial^2 A_z}{\partial \phi \partial z}, \\
 E_z &= j \frac{1}{\omega \varepsilon_z \mu_\phi} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial A_z}{\partial r} \right) + j \frac{1}{\omega \varepsilon_z} \frac{1}{r^2} \frac{\partial}{\partial \phi} \left(\frac{1}{\mu_r} \frac{\partial A_z}{\partial \phi} \right).
 \end{aligned}$$

In the above results for the TM^z mode in the cylindrical coordinate, the extra term $\varepsilon_r [(\mu_r - \mu_\phi) \frac{\partial}{\partial \phi} \frac{1}{\mu_r} + \frac{1}{\mu_r} \frac{\partial \mu_r}{\partial \phi}] \frac{1}{r^2} \frac{\partial A_z}{\partial \phi}$ comes from that the μ_r and μ_ϕ are ϕ dependent which can also be eliminated if μ_r and μ_ϕ are ϕ independent or $\mu_x = \mu_y$.

(C) TEM Modes:

The TEM modes can be solved by using either vector potential F or vector potential A . Firstly, consider the rectangular coordinate. Starting with the TEM^y mode, where $E_y = H_y = 0$. By using the vector potential F to solve the TEM^y mode, from equations (15), (20), and (21), we have,

$$H_y = -\frac{\partial \phi_m}{\partial y} - j\omega F_y = \frac{1}{j\omega \mu_x \varepsilon_z} \frac{\partial^2 F_y}{\partial y^2} - j\omega F_y = 0 \quad (27)$$

then we can have scalar wave equation of F_y ,

$$\frac{\partial^2 F_y}{\partial y^2} + \beta^2 F_y = 0; \quad \beta^2 = \omega^2 \mu_x \varepsilon_z = \omega^2 \mu_z \varepsilon_x \quad (28)$$

With a similar procedure, we can have the scalar wave equation of A_y ,

$$\frac{\partial^2 A_y}{\partial y^2} + \beta^2 A_y = 0; \quad \beta^2 = \omega^2 \mu_x \varepsilon_z = \omega^2 \mu_z \varepsilon_x \quad (29)$$

In summary the electric and magnetic fields of TEM^y mode are

$$\begin{aligned} F &= \hat{a}_y F_y(x, y, z); \\ \frac{\partial^2 F_y}{\partial y^2} + \beta^2 F_y &= 0, \quad \beta^2 = \omega^2 \mu_x \varepsilon_z = \omega^2 \mu_z \varepsilon_x; \\ \frac{\mu_x}{\varepsilon_x} &= \frac{\mu_z}{\varepsilon_z}; \\ E_x &= \frac{1}{\varepsilon_x} \frac{\partial F_y}{\partial z}, \quad E_y = 0, \quad E_z = -\frac{1}{\varepsilon_z} \frac{\partial F_y}{\partial x}; \\ H_x &= -j \frac{1}{\omega \mu_x \varepsilon_z} \frac{\partial^2 F_y}{\partial x \partial y}, \quad H_y = 0, \quad H_z = -j \frac{1}{\omega \mu_x \varepsilon_z} \frac{\partial^2 F_y}{\partial z \partial y}. \end{aligned}$$

or

$$\begin{aligned} A &= \hat{a}_y A_y(x, y, z); \\ \frac{\partial^2 A_y}{\partial y^2} + \beta^2 A_y &= 0, \quad \beta^2 = \omega^2 \mu_x \varepsilon_z = \omega^2 \mu_z \varepsilon_x; \\ \frac{\mu_x}{\varepsilon_x} &= \frac{\mu_z}{\varepsilon_z}; \\ H_x &= -\frac{1}{\mu_x} \frac{\partial A_y}{\partial z}, \quad H_y = 0, \quad H_z = \frac{1}{\mu_z} \frac{\partial A_y}{\partial x}; \\ E_x &= -j \frac{1}{\omega \mu_x \varepsilon_z} \frac{\partial^2 A_y}{\partial x \partial y}, \quad E_y = 0, \quad E_z = -j \frac{1}{\omega \mu_x \varepsilon_z} \frac{\partial^2 A_y}{\partial z \partial y}. \end{aligned}$$

The results for TEM^x and TEM^z modes are also in the appendix for reference.

In the cylindrical coordinate for TEM^z mode:

$$\begin{aligned}
F &= \hat{a}_z F_z(r, \phi, z); \\
\frac{\partial^2 F_z}{\partial z^2} + \beta^2 F_z &= 0, \quad \beta^2 = \omega^2 \mu_r \varepsilon_\phi = \omega^2 \mu_\phi \varepsilon_r; \\
\frac{\mu_r}{\varepsilon_r} &= \frac{\mu_\phi}{\varepsilon_\phi}; \\
E_r &= -\frac{1}{\varepsilon_r} \frac{1}{r} \frac{\partial F_z}{\partial \phi}, \quad E_\phi = \frac{1}{\varepsilon_\phi} \frac{\partial F_z}{\partial r}, \quad E_z = 0; \\
H_r &= -j \frac{1}{\omega \mu_r \varepsilon_\phi} \frac{\partial^2 F_z}{\partial r \partial z}, \quad H_\phi = -j \frac{1}{\omega \mu_\phi \varepsilon_r} \frac{1}{r} \frac{\partial^2 F_z}{\partial \phi \partial z}, \quad H_z = 0.
\end{aligned}$$

or

$$\begin{aligned}
A &= \hat{a}_z A_z(r, \phi, z); \\
\frac{\partial^2 A_z}{\partial z^2} + \beta^2 A_z &= 0, \quad \beta^2 = \omega^2 \mu_r \varepsilon_\phi = \omega^2 \mu_\phi \varepsilon_r; \\
\frac{\mu_r}{\varepsilon_r} &= \frac{\mu_\phi}{\varepsilon_\phi}; \\
H_r &= \frac{1}{\mu_r} \frac{1}{r} \frac{\partial A_z}{\partial \phi}, \quad H_\phi = -\frac{1}{\mu_\phi} \frac{\partial A_z}{\partial r}, \quad H_z = 0; \\
E_r &= -j \frac{1}{\omega \varepsilon_r \mu_\phi} \frac{\partial^2 A_z}{\partial r \partial z}, \quad E_\phi = -j \frac{1}{\omega \varepsilon_\phi \mu_r} \frac{1}{r} \frac{\partial^2 A_z}{\partial \phi \partial z}, \quad E_z = 0.
\end{aligned}$$

3. SUMMARY

This investigation presents the method for solving the field configurations of TE, TM, and TEM modes for an biaxial anisotropic medium. The concise scalar wave equations of vector potential for various modes and directions are obtained in both rectangular and cylindrical coordinates. Compared to those scalar wave equations of the rectangular coordinate, for the cylindrical coordinate, the scalar wave equations of vector potentials F_z and A_z for both TE^z and TM^z modes display an extra term, from the ϕ dependent of permittivity and permeability respectively. Those extra terms can be eliminated in a uniaxial medium. By solving those scalar wave equations, the electric and magnetic fields of various modes can be obtained by adding adequate boundary conditions. The existence of TE, TM, and TEM modes is clarified. The relationship between permittivity and permeability is critical factor for the existence of desired modes.

APPENDIX A.

1. The equations for TE^x and TE^z modes in the rectangular coordinate:

For TE^x mode:

$$\begin{aligned}
 F &= \hat{a}_x F_x(x, y, z); \\
 \mu_x \frac{\partial^2 F_x}{\partial x^2} + \mu_y \frac{\partial^2 F_x}{\partial y^2} + \mu_z \frac{\partial^2 F_x}{\partial z^2} + \kappa^2 F_x &= 0, \quad \kappa^2 = \omega^2 \mu_x \varepsilon_y \mu_z = \omega^2 \mu_x \varepsilon_z \mu_y; \\
 \frac{\mu_y}{\varepsilon_y} &= \frac{\mu_z}{\varepsilon_z}; \\
 E_x &= 0, \quad E_y = -\frac{1}{\varepsilon_y} \frac{\partial F_x}{\partial z}, \quad E_z = \frac{1}{\varepsilon_z} \frac{\partial F_x}{\partial y}; \\
 H_x &= j \frac{1}{\omega \mu_x \varepsilon_z} \frac{\partial^2 F_x}{\partial y^2} + j \frac{1}{\omega \mu_x \varepsilon_y} \frac{\partial^2 F_x}{\partial z^2}, \quad H_y = -j \frac{1}{\omega \mu_y \varepsilon_z} \frac{\partial^2 F_x}{\partial y \partial x}, \\
 H_z &= -j \frac{1}{\omega \mu_z \varepsilon_y} \frac{\partial^2 F_x}{\partial z \partial x}
 \end{aligned}$$

For TE^z mode:

$$\begin{aligned}
 F &= \hat{a}_z F_z(x, y, z); \\
 \mu_x \frac{\partial^2 F_z}{\partial x^2} + \mu_y \frac{\partial^2 F_z}{\partial y^2} + \mu_z \frac{\partial^2 F_z}{\partial z^2} + \kappa^2 F_z &= 0, \quad \kappa^2 = \omega^2 \mu_z \varepsilon_x \mu_y = \omega^2 \mu_z \varepsilon_y \mu_x; \\
 \frac{\mu_x}{\varepsilon_x} &= \frac{\mu_y}{\varepsilon_y}; \\
 E_x &= -\frac{1}{\varepsilon_x} \frac{\partial F_z}{\partial y}, \quad E_y = \frac{1}{\varepsilon_y} \frac{\partial F_z}{\partial x}, \quad E_z = 0; \\
 H_x &= -j \frac{1}{\omega \mu_x \varepsilon_y} \frac{\partial^2 F_z}{\partial x \partial z}, \quad H_y = -j \frac{1}{\omega \mu_y \varepsilon_x} \frac{\partial^2 F_z}{\partial y \partial z}, \\
 H_z &= j \frac{1}{\omega \mu_z \varepsilon_y} \frac{\partial^2 F_z}{\partial x} + j \frac{1}{\omega \mu_z \varepsilon_x} \frac{\partial^2 F_z}{\partial y^2}
 \end{aligned}$$

2. The equations for TM^x and TM^z modes in the rectangular coordinate:

For TM^x mode:

$$\begin{aligned}
 A &= \hat{a}_x A_x(x, y, z); \\
 \varepsilon_x \frac{\partial^2 A_x}{\partial x^2} + \varepsilon_y \frac{\partial^2 A_x}{\partial y^2} + \varepsilon_z \frac{\partial^2 A_x}{\partial z^2} + \kappa^2 A_x &= 0, \quad \kappa^2 = \omega^2 \varepsilon_x \varepsilon_y \mu_z = \omega^2 \varepsilon_x \varepsilon_z \mu_y; \\
 \frac{\mu_y}{\varepsilon_y} &= \frac{\mu_z}{\varepsilon_z}; \\
 H_x &= 0, \quad H_y = \frac{1}{\mu_y} \frac{\partial A_x}{\partial z}, \quad H_z = -\frac{1}{\mu_z} \frac{\partial A_x}{\partial y}; \\
 E_x &= j \frac{1}{\omega \varepsilon_x \mu_z} \frac{\partial^2 A_x}{\partial y^2} + j \frac{1}{\omega \varepsilon_x \mu_y} \frac{\partial^2 A_x}{\partial z^2}, \quad E_y = -j \frac{1}{\omega \varepsilon_y \mu_z} \frac{\partial^2 A_x}{\partial y \partial x}, \\
 E_z &= -j \frac{1}{\omega \varepsilon_z \mu_y} \frac{\partial^2 A_x}{\partial z \partial x}
 \end{aligned}$$

For TM^z mode:

$$\begin{aligned}
 A &= \hat{a}_z A_z(x, y, z); \\
 \varepsilon_x \frac{\partial^2 A_z}{\partial x^2} + \varepsilon_y \frac{\partial^2 A_z}{\partial y^2} + \varepsilon_z \frac{\partial^2 A_z}{\partial z^2} + \kappa^2 A_z &= 0, \quad \kappa^2 = \omega^2 \varepsilon_z \varepsilon_x \mu_y = \omega^2 \varepsilon_z \varepsilon_y \mu_x; \\
 \frac{\mu_x}{\varepsilon_x} &= \frac{\mu_y}{\varepsilon_y}; \\
 H_x &= \frac{1}{\mu_x} \frac{\partial A_z}{\partial y}, \quad H_y = -\frac{1}{\mu_y} \frac{\partial A_z}{\partial x}, \quad H_z = 0; \\
 E_x &= -j \frac{1}{\omega \varepsilon_x \mu_y} \frac{\partial^2 A_z}{\partial x \partial z}, \quad E_y = -j \frac{1}{\omega \varepsilon_y \mu_x} \frac{\partial^2 A_z}{\partial y \partial z}, \\
 E_z &= j \frac{1}{\omega \varepsilon_z \mu_y} \frac{\partial^2 A_z}{\partial x^2} + j \frac{1}{\omega \varepsilon_z \mu_x} \frac{\partial^2 A_z}{\partial y^2}
 \end{aligned}$$

3. The equations for TEM^x and TEM^z modes in the rectangular coordinate:

For TEM^x mode:

$$\begin{aligned}
 F &= \hat{a}_x F_x(x, y, z); \\
 \frac{\partial^2 F_x}{\partial x^2} + \beta^2 F_x &= 0, \quad \beta^2 = \omega^2 \mu_y \varepsilon_z = \omega^2 \mu_z \varepsilon_y; \\
 \frac{\mu_y}{\varepsilon_y} &= \frac{\mu_z}{\varepsilon_z}; \\
 E_x &= 0, \quad E_y = -\frac{1}{\varepsilon_y} \frac{\partial F_x}{\partial z}, \quad E_z = \frac{1}{\varepsilon_z} \frac{\partial F_x}{\partial y}; \\
 H_x &= 0, \quad H_y = -j \frac{1}{\omega \mu_y \varepsilon_z} \frac{\partial^2 F_x}{\partial y \partial x}, \quad H_z = -j \frac{1}{\omega \mu_y \varepsilon_z} \frac{\partial^2 F_x}{\partial z \partial x}.
 \end{aligned}$$

or

$$\begin{aligned}
 A &= \hat{a}_x A_x(x, y, z); \\
 \frac{\partial^2 A_x}{\partial x^2} + \beta^2 A_x &= 0, \quad \beta^2 = \omega^2 \mu_y \varepsilon_z = \omega^2 \mu_z \varepsilon_y; \\
 \frac{\mu_y}{\varepsilon_y} &= \frac{\mu_z}{\varepsilon_z}; \\
 H_x &= 0, \quad H_y = \frac{1}{\mu_y} \frac{\partial A_x}{\partial z}, \quad H_z = -\frac{1}{\mu_z} \frac{\partial A_x}{\partial y}; \\
 E_x &= 0, \quad E_y = -j \frac{1}{\omega \mu_y \varepsilon_z} \frac{\partial^2 A_x}{\partial y \partial x}, \quad E_z = -j \frac{1}{\omega \mu_y \varepsilon_z} \frac{\partial^2 A_x}{\partial z \partial x}.
 \end{aligned}$$

For TEM^z mode:

$$\begin{aligned}
 F &= \hat{a}_z F_z(x, y, z); \\
 \frac{\partial^2 F_z}{\partial z^2} + \beta^2 F_z &= 0, \quad \beta^2 = \omega^2 \mu_x \varepsilon_y = \omega^2 \mu_y \varepsilon_x; \\
 \frac{\mu_x}{\varepsilon_x} &= \frac{\mu_y}{\varepsilon_y}; \\
 E_x &= -\frac{1}{\varepsilon_x} \frac{\partial F_z}{\partial y}, \quad E_y = \frac{1}{\varepsilon_y} \frac{\partial F_z}{\partial x}, \quad E_z = 0; \\
 H_x &= -j \frac{1}{\omega \mu_x \varepsilon_y} \frac{\partial^2 F_z}{\partial x \partial z}, \quad H_y = -j \frac{1}{\omega \mu_x \varepsilon_y} \frac{\partial^2 F_z}{\partial y \partial z}, \quad H_z = 0.
 \end{aligned}$$

or

$$\begin{aligned}
 A &= \hat{a}_z A_z(x, y, z); \\
 \frac{\partial^2 A_z}{\partial z^2} + \beta^2 A_z &= 0, \quad \beta^2 = \omega^2 \mu_x \varepsilon_y = \omega^2 \mu_y \varepsilon_x; \\
 \frac{\mu_x}{\varepsilon_x} &= \frac{\mu_y}{\varepsilon_y}; \\
 H_x &= \frac{1}{\mu_x} \frac{\partial A_z}{\partial y}, \quad H_y = -\frac{1}{\mu_y} \frac{\partial A_z}{\partial x}, \quad H_z = 0; \\
 E_x &= -j \frac{1}{\omega \mu_x \varepsilon_y} \frac{\partial^2 A_z}{\partial x \partial z}, \quad E_y = -j \frac{1}{\omega \mu_x \varepsilon_y} \frac{\partial^2 A_z}{\partial y \partial z}, \quad E_z = 0.
 \end{aligned}$$

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REFERENCES

1. Moss, C. D., F. L. Teixeira, and J. A. Kong, "Analysis and compensation of numerical dispersion in the FDTD method for layered, anisotropic media," *IEEE Trans. Antennas Propagat.*, Vol. 50, 1174–1184, Sep. 2002.
2. Barrowes, B. E., C. O. Ao, F. L. Teixeira, and J. A. Kong, "Sparse matrix/canonical grid method applied to 3-D dense medium simulations," *IEEE Trans. Antennas Propagat.*, Vol. 51, 48–58, Jan. 2003.
3. Krowne, C. M., "Spectral-domain determination of the propagation constant in biaxial planar media," *Int. J. Electronics*, Vol. 59, 315–332, 1985.
4. Van Trier, A. A. and M. Th., "Guided electromagnetic waves in anisotropic media," *Appl. Sci. Res. (Section B)*, Vol. 3, 305–371, 1953.
5. Akinin, N., A. El Moussaoui, and M. Essaaidi, "Rigorous analysis of shielded uniaxial anisotropic DRs by the mode-matching method," *Microwave Opt. Technol. Lett.*, Vol. 19, 233–238, June 1995.

6. Palanichamy, S., A. Biswas, and R. K. Shukla, "Full-wave analysis of dominant and higher order modes in microstrip-slot coupled line with uniaxial anisotropic dielectric substrates," *Microwave Opt. Technol. Lett.*, Vol. 25, 400–406, June 2000.
7. Liu, S., L. W. Li, M. S. Leong, and T. S. Yeo, "Field representation in general rotationally uniaxial anisotropic media using spherical vector wave functions," *Microwave Opt. Technol. Lett.*, Vol. 25, 159–162, May 2000.
8. Li, L. W., S. H. Liu, M. S. Leong, and T. S. Yeo, "Circular cylindrical waveguide filled with uniaxial anisotropic media-electromagnetic fields and dyadic Green's functions," *IEEE Trans. Microwave Theory Tech.*, Vol. 49, 1361–1364, July 2001.
9. Wang, C. and K. A. Zaki, "Generalized multiplayer anisotropic dielectric resonators," *IEEE Trans. Microwave Theory Tech.*, Vol. 48, 60–66, Jan. 2001.
10. Losada, V., R. R. Boix, and M. Horno, "Full-wave analysis of circular microstrip resonators in multilayered media containing uniaxial anisotropic dielectrics, magnetized ferrites, and chiral materials," *IEEE Trans. Microwave Theory Tech.*, Vol. 48, 1057–1064, June 2000.
11. Lindell, I. V. and F. Olyslager, "Decomposition of electromagnetic sources in axially chiral uniaxial anisotropic media," *J. Electromagnetic Waves Applications*, Vol. 10, 51–59, 1996.
12. Olyslager, F. and D. D. Zutter, "Rigorous boundary integral equation solution for general isotropic and uniaxial anisotropic dielectric waveguides in multilayered media including losses, gain and leakage," *IEEE Trans. Microwave Theory Tech.*, Vol. 41, 1385–1392, Aug. 1993.
13. Shin, S. C. and C. S. Kim, "A new method for analyzing torque curves of uniaxial anisotropic materials," *IEEE Trans. Magnetism*, Vol. 27, 4852–4854, Nov. 1991.
14. Lindell, I. V., "TE/TM decomposition of electromagnetic sources in uniaxial anisotropic media," *Microwave Opt. Technol. Lett.*, Vol. 19, 233–238, June 1995.
15. Balanis, C. A., *Advanced Engineering Electromagnetics*, Chap. 6, New York, John Wiley & Sons, 1989.

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